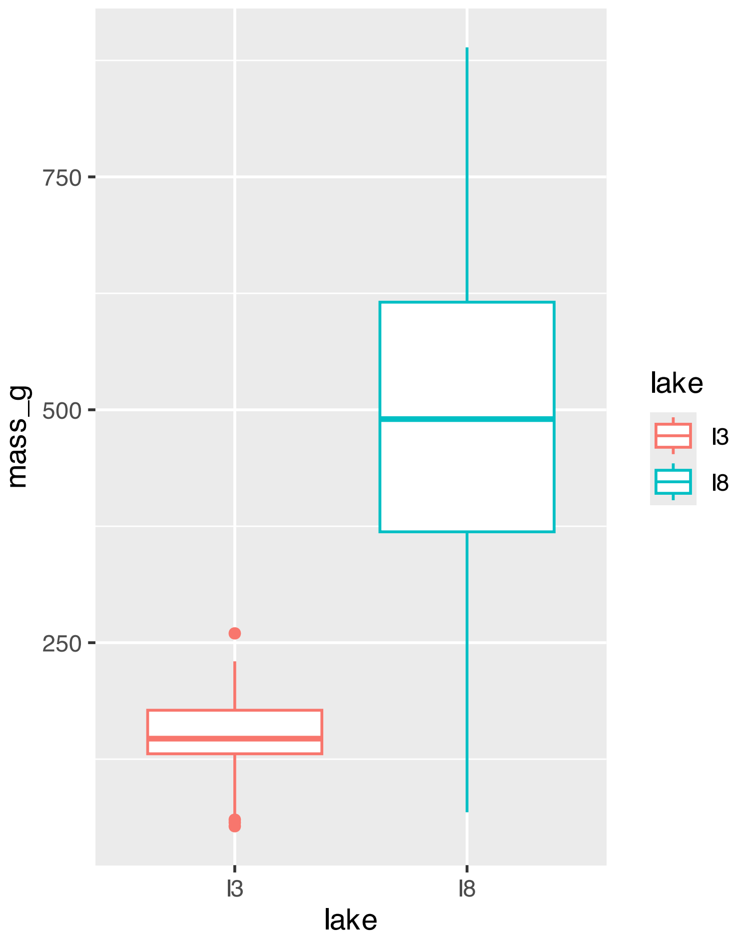
Lecture 09 Correlation and Regression

Bill Perry

Lecture 8: Review

Covered

- Study design
- Causality in ecology
- Experimental design:
  - Replication, controls, randomization, independence
- Sampling in field studies
- Power analysis: *a priori* and *post hoc*
- Study design and analysis



Lecture 9: Overview

The objectives:

This lecture covers two fundamental statistical techniques in biology: correlation and regression analysis. Based on Chapters 16-17 from Whitlock & Schluter's *The Analysis of Biological Data* (3rd edition), we'll explore:

- Correlation analysis: measuring relationships between variables

- The distinction between correlation and regression
- Simple linear regression: predicting one variable from another
- Estimating and interpreting regression parameters
- Testing assumptions and handling violations
- Analysis of variance in regression
- Model selection and comparison

Lecture 9: Correlation vs. Regression:

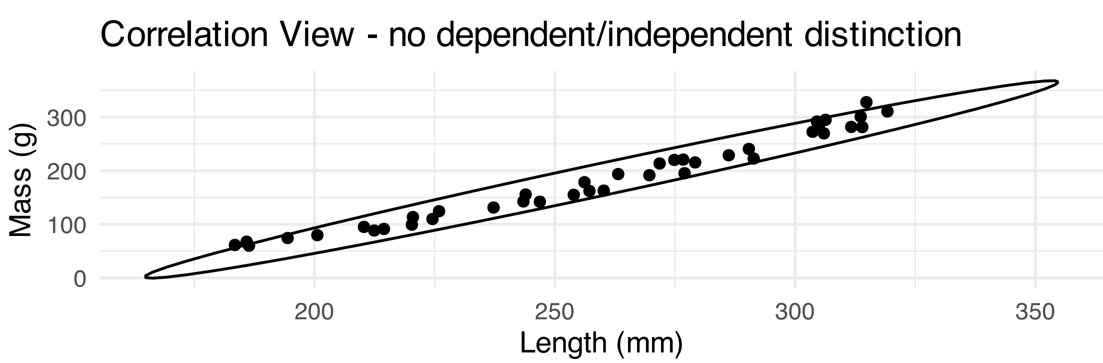
What's the Difference?

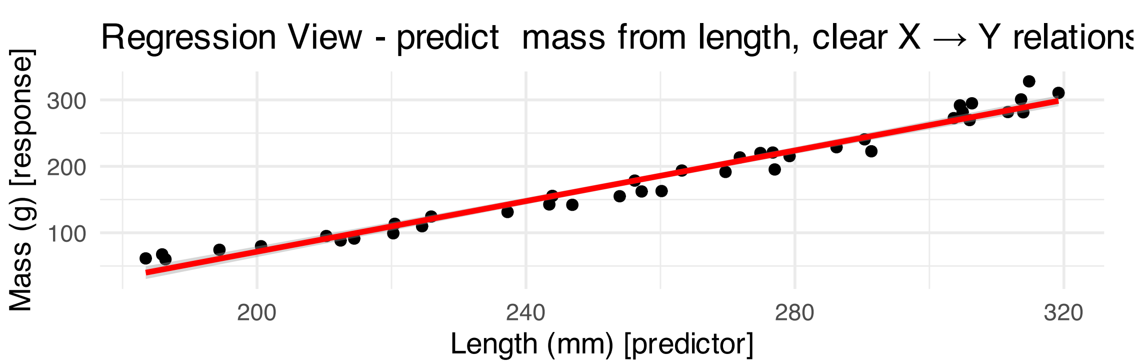
Correlation Analysis:

- Measures the strength and direction of a relationship between two numerical variables
- Both X and Y are random variables (both measured, neither manipulated)
- Variables are typically on equal footing (either could be X or Y)
- No cause-effect relationship implied
- Quantifies the degree to which variables are related
- Expressed as a correlation coefficient (r) from −1 to +1

Regression Analysis:

- Predicts one variable (Y) from another (X)
- X is often fixed or controlled (manipulated)
- Y is the response variable of interest
- Often implies a cause-effect relationship
- Produces an equation for prediction
- Estimates slope and intercept parameters





Lecture 9: Correlation Analysis

What Is Correlation?

Correlation analysis measures the strength and direction of a relationship between two numerical variables:

- Ranges from −1 to +1
- +1 indicates perfect positive correlation
- 0 indicates no correlation
- −1 indicates perfect negative correlation

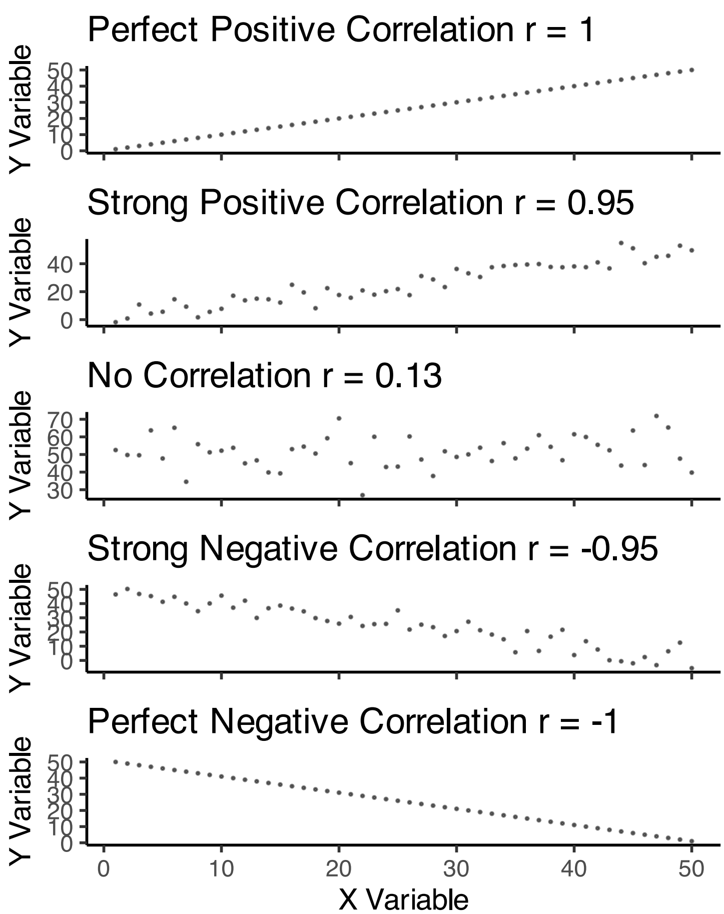
The **Pearson correlation coefficient (r)** is defined as:

$$r = \frac{\sum_i \left(X_i - X^{-}\right)\left(Y_i - Y^{-}\right)}{\sqrt{\sum_i \left(X_i - X^{-}\right)^2 \sum_i \left(Y_i - Y^{-}\right)^2}}$$

This can be simplified as:

$$r = \frac{\text{Covariance }(X, Y)}{s_X \cdot s_Y}$$

Where s_X and s_Y are the standard deviations of X and Y.



Lecture 9: Correlation Analysis

Example 16.1: Flipping the Bird

Nazca boobies (*Sula granti*) - Do aggressive behaviors as a chick predict future aggressive behavior as an adult?

- correlation is r = 0.534 - moderate positive relationship
- p-value = 0.007 correlation is statistically significant.

For a Pearson correlation coefficient (r) of 0.53372:

- This is r (not rho as Spearman nonparticipant below), as indicated by "cor" in your output
- To determine the amount of variation explained, you square this value: $r^2 = 0.53372^2 = 0.2849$ (or approximately 28.49%)
- means about 28.49% of the variance in one variable can be explained by the other variable

Note $\mathrm{t} = \frac{r}{SE_r}$

```
[1] 0.5337225
```

```

	Pearson's product-moment correlation

data:  booby_data$visits_as_nestling and booby_data$future_aggression
t = 2.9603, df = 22, p-value = 0.007229
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.1660840 0.7710999
sample estimates:
      cor 
0.5337225 
```

Lecture 9: Correlation Analysis

Example 16.1: Flipping the Bird

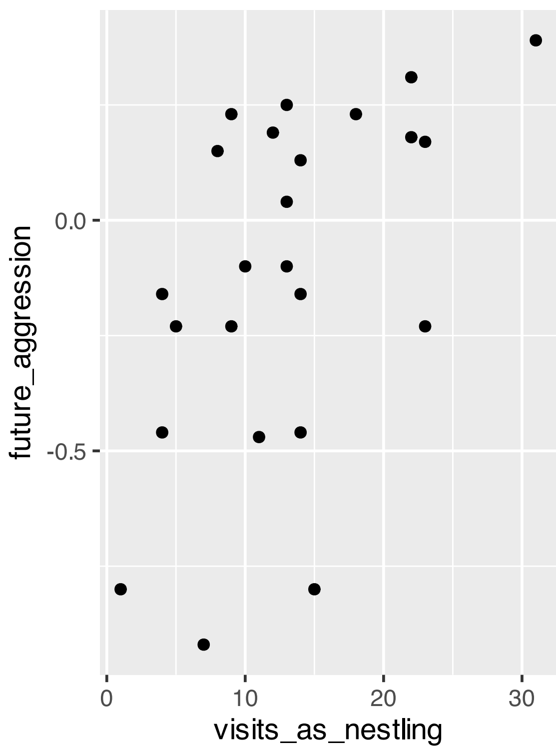
Interpretation: The correlation coefficient of r = 0.534 suggests that Nazca boobies who experienced more visits from non-parent adults as nestlings tend to display more aggressive behavior as adults. This supports the hypothesis that early experiences influence adult behavior patterns in this species.

Standard Error:

$\mathrm{SE}_r = \sqrt{\frac{1-r^2}{n-2}}$

SE = 0.180

Need to be sure relationship is not curved - note below



Lecture 9: Correlation Analysis

Testing Assumptions for Correlation

As described in Section 16.3, correlation analysis has key assumptions:

1. **Random sampling**: Observations should be a random sample from the population
2. **Bivariate normality**: Both variables follow a normal distribution, and their joint distribution is bivariate normal
3. **Linear relationship**: The relationship between variables is linear, not curved

Let's check these assumptions using the lion data from Example 17.1 Lion Noses:

```
	Shapiro-Wilk normality test

data:  lion_data$proportion_black
W = 0.88895, p-value = 0.003279
```

```
	Shapiro-Wilk normality test

data:  lion_data$age_years
W = 0.87615, p-value = 0.001615
```

Lecture 9: Correlation Analysis

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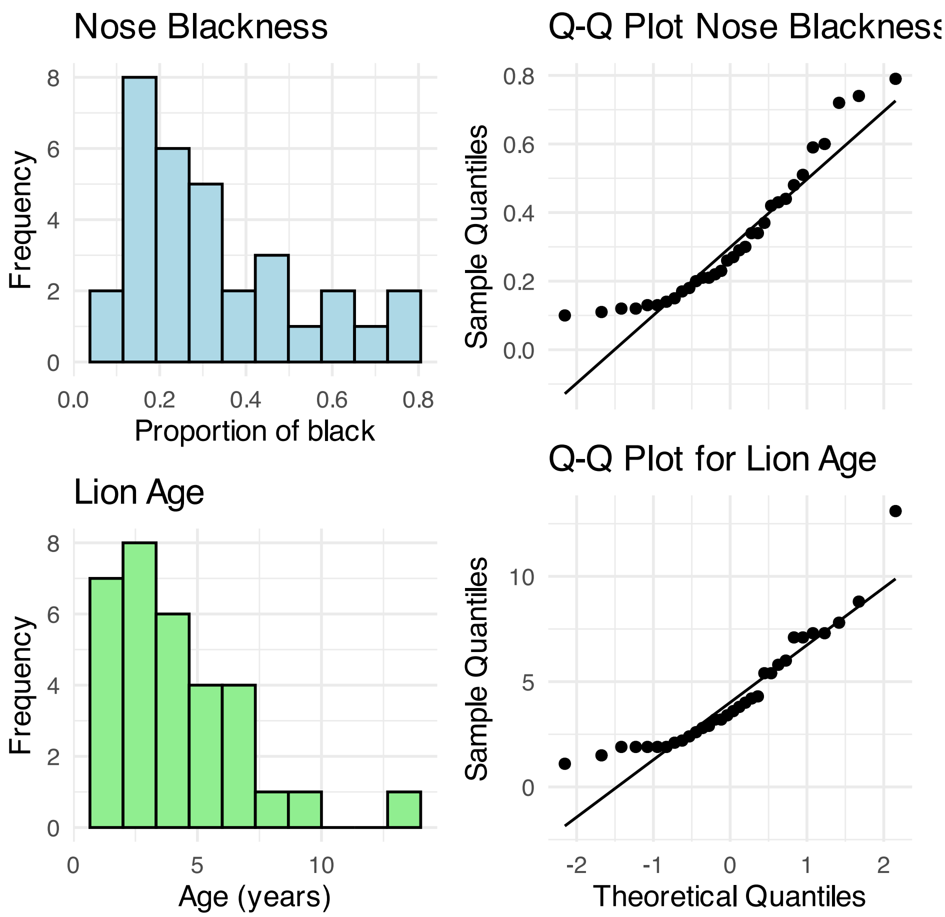


Lecture 9: Correlation Analysis

What to do if assumptions are violated:
Transform one or both variables (log, square root, etc.)

Use non-parametric correlation (**Spearman's rank correlation**) or Kendall's tau ?

Examine the data for outliers or influential points

To understand the amount of variation explained, you can square the Spearman's rho value.

For your value of 0.74485:

$\rho^2 = 0.74485^2 = 0.5548$

This means approximately 55.48% of the variance in ranks of one variable can be explained by the ranks of the other variable. This is similar to how R^2 works in linear regression, but specifically for ranked data.

```

	Spearman's rank correlation rho

data:  lion_data$proportion_black and lion_data$age_years
S = 1392.1, p-value = 1.013e-06
alternative hypothesis: true rho is not equal to 0
sample estimates:
```

```
      rho
0.7448561
```

Lecture 9: Correlation Analysis

Correlation: Important Considerations

The correlation coefficient depends on the range

- Restricting range of values can reduce the correlation coefficient
- Comparing correlations between studies requires similar ranges of values

Measurement error affects correlation

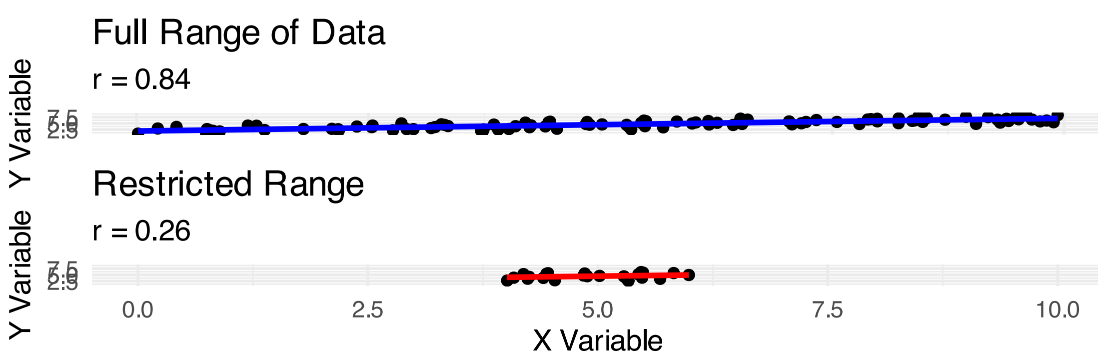
- Measurement error in X or Y tends to weaken observed correlation
- This bias is called **attenuation**
- True correlation typically stronger than observed correlation

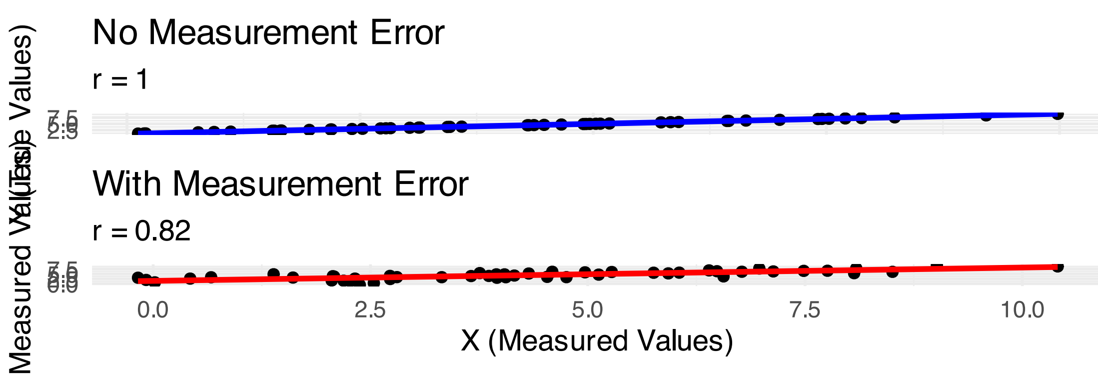
Correlation vs. Causation

- Correlation does not imply causation
- Three possible explanations for correlation:
  1. X causes Y
  2. Y causes X
  3. Z (a third variable) causes both X and Y

Correlation significance test

- H_0: $\rho = 0$ (no correlation in population)
- H_1: $\rho \neq 0$ (correlation exists in population)
- **Test statistic: $t = r / SE(r)$ with $df = n-2$**





Lecture 9: Linear Regression

Simple Linear Regression Model

Simple linear regression models the relationship between a response variable (Y) and a predictor variable (X).

The **population** regression model

$$Y = \alpha + \beta X + \varepsilon$$

Where:

- Y is the response variable
- X is the predictor variable
- α (alpha) is the intercept (value of Y when X=0)
- β (beta) is the slope (change in Y per unit change in X)
- ε (epsilon) is the error term (random deviation from the line)

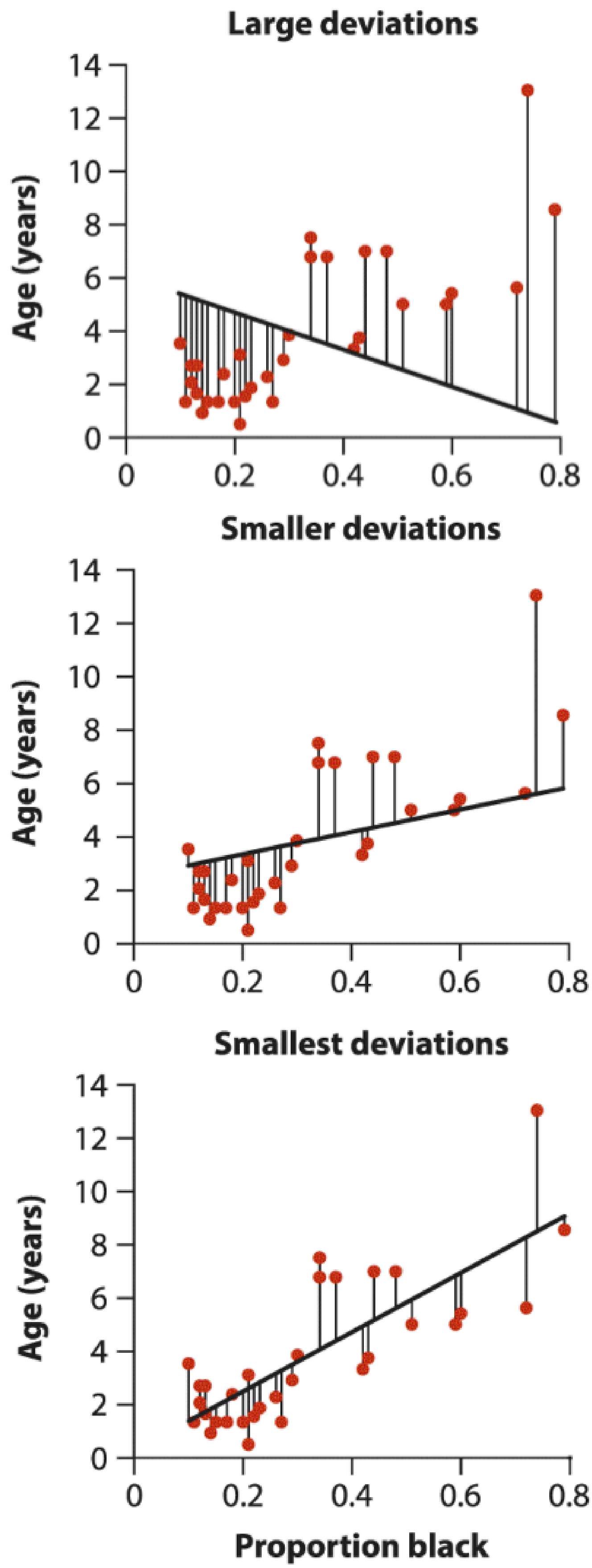
The **sample** regression equation is:

$$\hat{Y} = a + bX$$

Where:

- $\hat{Y}$ is the predicted value of Y
- a is the estimate of α (intercept)
- b is the estimate of β (slope)

Method of Least Squares: The line is chosen to minimize the sum of squared vertical distances (residuals) between observed and predicted Y values.



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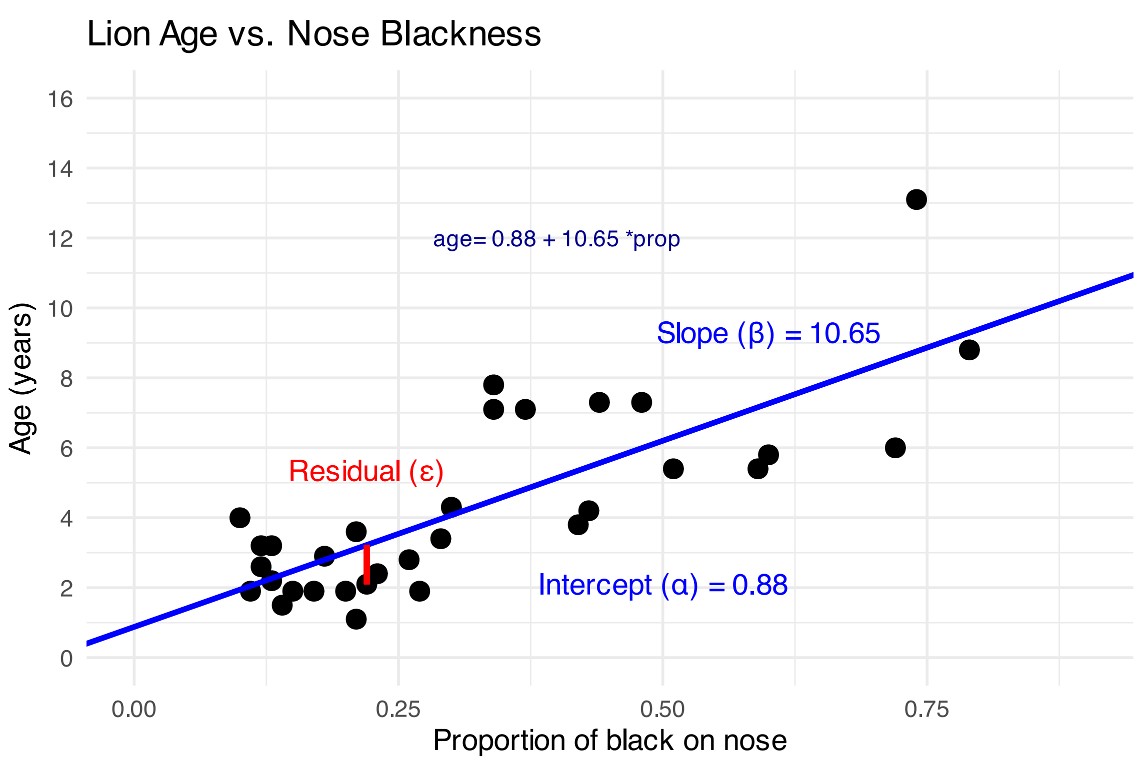
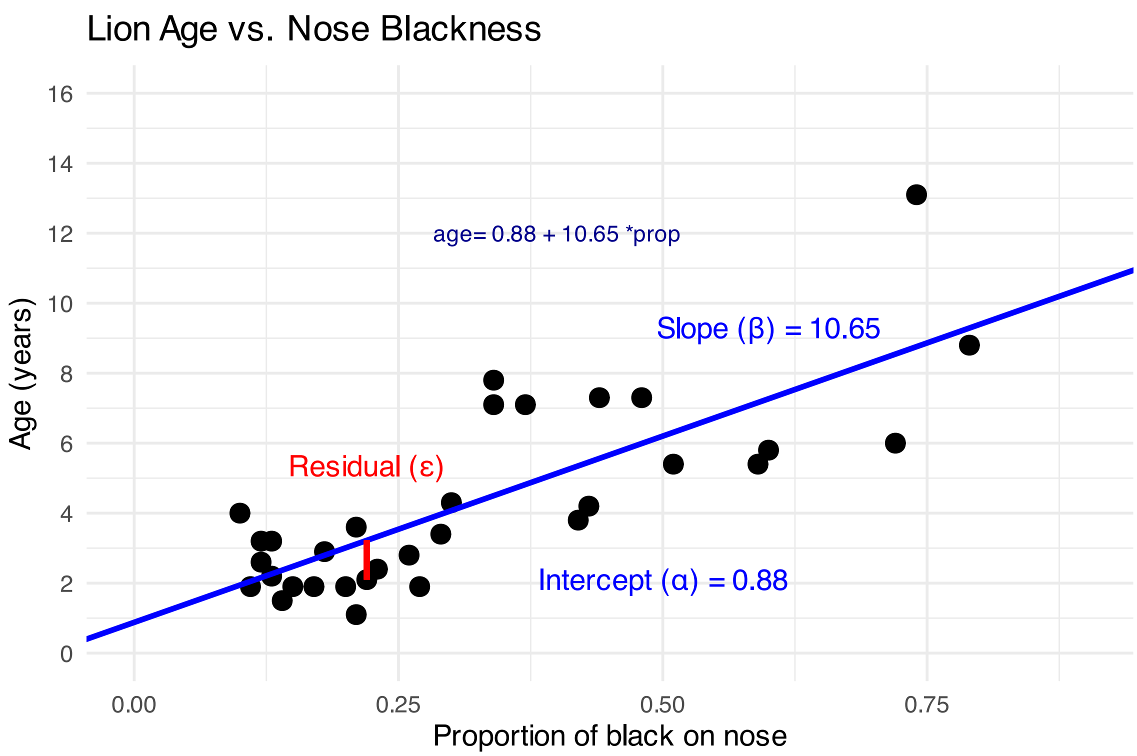


Lecture 9: Linear Regression

From Example 17.1 in the textbook the regression line for the lion data is:

$\text{age} = 0.88 + 10.65 \times \text{proportion}_{black}$

This means: - When a lion has no black on its nose (proportion = 0), its predicted age is 0.88 years - For each 0.1 increase in the proportion of black, age increases by 1.065 years - The slope (10.65) indicates that lions with more black on their noses tend to be older



Lecture 9: Linear Regression

Simple Linear Regression Model

- male lions develop more black pigmentation on their noses as they age.
- can be used to estimate the age of lions in the field.

```
Call:
lm(formula = age_years ~ proportion_black, data = lion_data)

Residuals:
    Min      1Q  Median      3Q     Max 
-2.5449 -1.1117 -0.5285  0.9635  4.3421 

Coefficients:
                 Estimate Std. Error t value Pr(>|t|)    
(Intercept)        0.8790     0.5688   1.545    0.133    
proportion_black  10.6471     1.5095   7.053 7.68e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.669 on 30 degrees of freedom
Multiple R-squared:  0.6238,	Adjusted R-squared:  0.6113 
F-statistic: 49.75 on 1 and 30 DF,  p-value: 7.677e-08
```

Lecture 9: Linear Regression

Simple Linear Regression Model

The calculation for slope (b) is:

$$b = \frac{\sum_i \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_i \left(X_i - \bar{X}\right)^2}$$

Given: - $\bar{X} = 0.3222$ - $\bar{Y} = 4.3094$ - $\sum_i \left(X_i - \bar{X}\right)^2 = 1.2221$ - $\sum_i \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) = 13.0123$

b = 13.0123 / 1.2221 = 10.647

Intercept (a): $a = \bar{Y} - b\bar{X} = 4.3094 - 10.647(0.3222) = 0.879$

Making predictions:

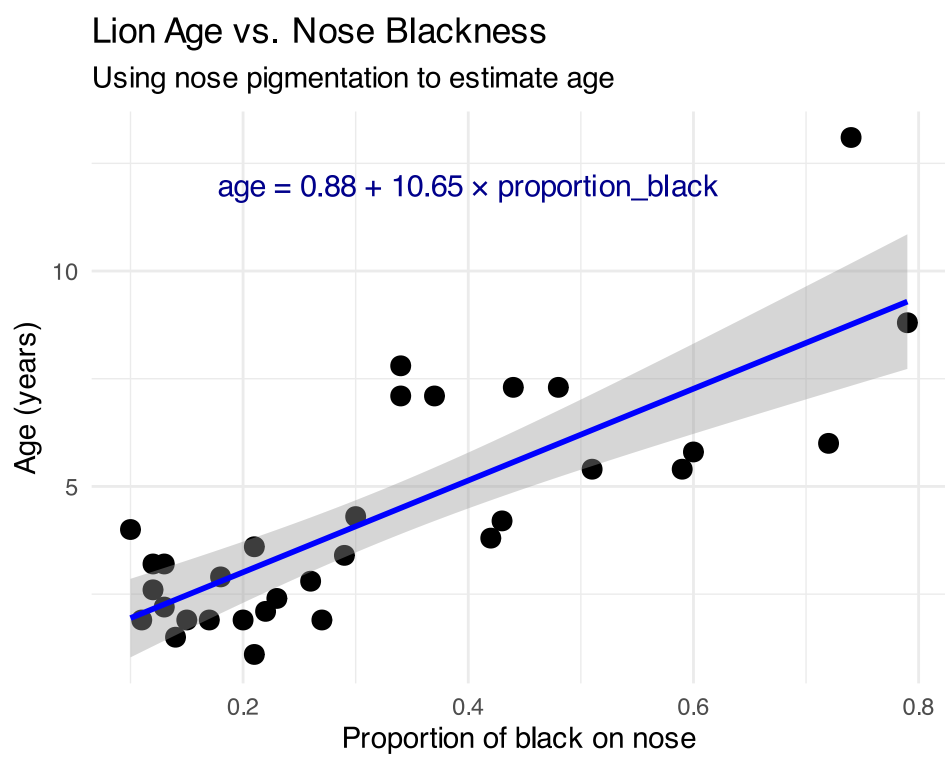
To predict the age of a lion with 0.50 proportion of black on its nose:

$$\hat{Y} = 0.88 + 10.65(0.50) = 6.2 \quad \text{years}$$

Confidence intervals vs. Prediction intervals:

- **Confidence interval**: Range for the mean age of all lions with 0.50 black
- **Prediction interval**: Range for an individual lion with 0.50 black

Both intervals are narrowest near $\bar{X}$ and widen as X moves away from the mean.



Lecture 9: Linear Regression

Example Prairie Home Companion

- Does biodiversity affect ecosystem stability?
- Tilman et al. (2006) investigated using experimental plots varying plant species

```
# A tibble: 6 × 2
  species_number log_stability
```

```
           <dbl>         <dbl>
1              1         0.763
2              1         1.45
3              1         1.51
4              1         0.747
5              1         0.983
6              1         1.12
```

```

Call:
lm(formula = log_stability ~ species_number, data = prairie_data)

Residuals:
     Min       1Q   Median       3Q      Max
-0.82774 -0.25344 -0.00426  0.27498  0.75240

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    1.252629   0.041023  30.535  < 2e-16 ***
species_number 0.025984   0.004926   5.275 4.28e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3433 on 159 degrees of freedom
Multiple R-squared:  0.149, Adjusted R-squared:  0.1436
F-statistic: 27.83 on 1 and 159 DF,  p-value: 4.276e-07
```

```
[1] "rsquared is:  0.148953385305455"
```

```
Analysis of Variance Table

Response: log_stability
                Df  Sum Sq Mean Sq F value    Pr(>F)
species_number   1  3.2792  3.2792  27.829 4.276e-07 ***
Residuals      159 18.7358  0.1178
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Lecture 9: Linear Regression

The hypothesis test asks whether the slope equals zero:

- H_0: $\beta = 0$ (species number does not affect stability)
- H_1: $\beta \neq 0$ (species number does affect stability)

The test statistic is: $t = \frac{b-\beta_0}{SE_b}$
With df = n - 2 = 161 - 2 = 159

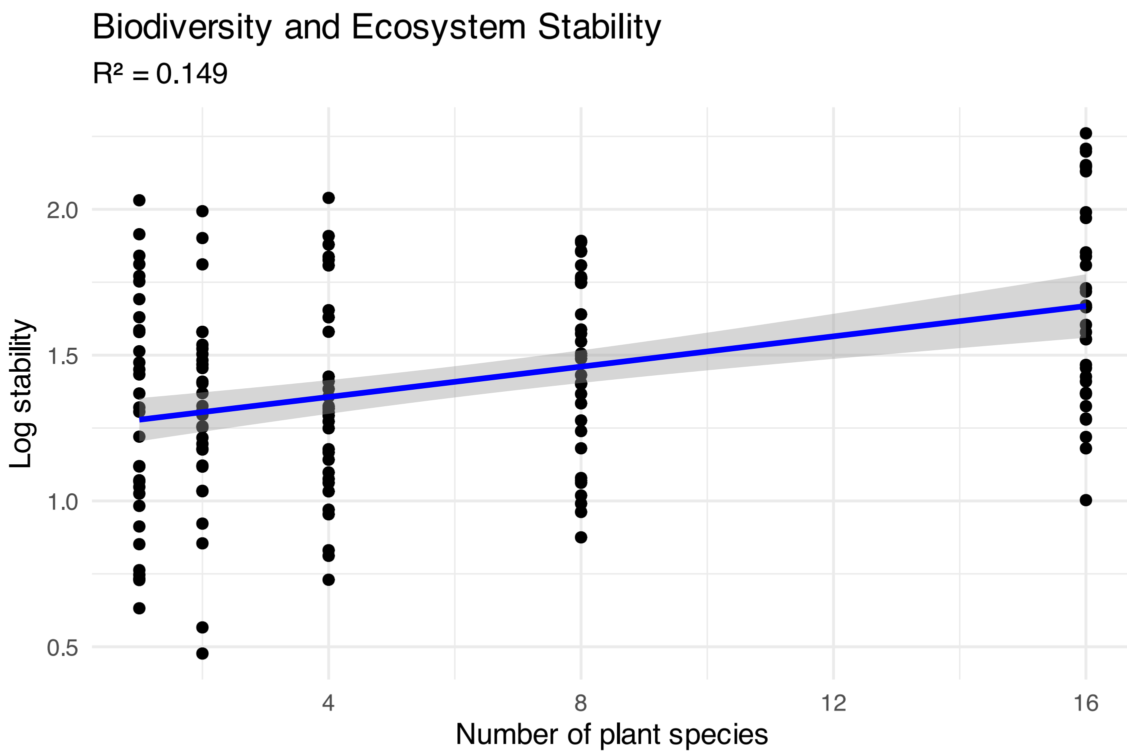
Interpretation:

The slope estimate is 0.033, indicating that log stability increases by 0.033 units for each additional plant species in the plot.

The p-value is very small (2.73e-10), providing strong evidence to reject the null hypothesis that species number has no effect on ecosystem stability.

$R^2 = 0.222$, meaning that approximately 22.2% of the variation in log stability is explained by the number of plant species.

This supports the biodiversity-stability hypothesis: more diverse plant communities have more stable biomass production over time.



Lecture 9: Linear Regression

Testing Regression Assumptions

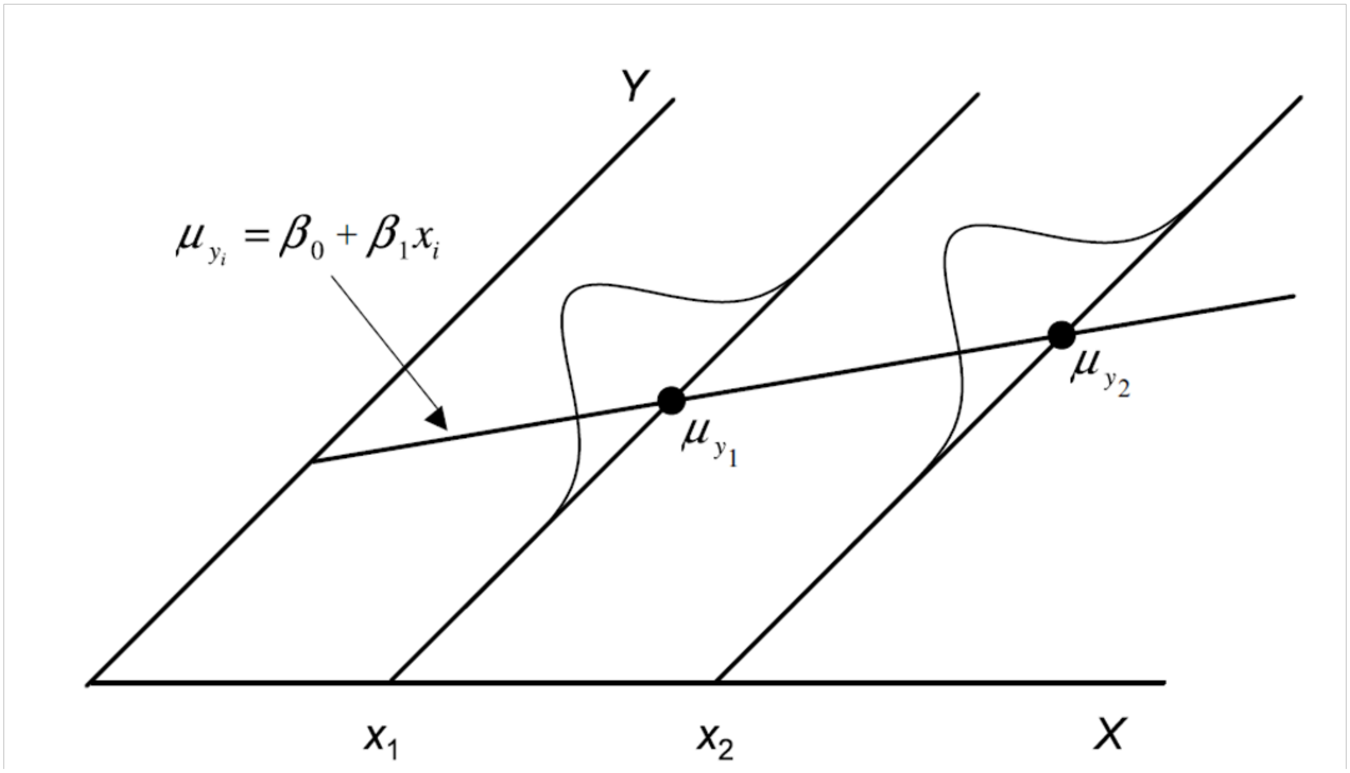
linear regression has four key assumptions:

1. **Linearity**: The relationship between X and Y is linear
2. **Independence**: Observations are independent
3. **Homoscedasticity**: Equal variance across all values of X
4. **Normality**: Residuals are normally distributed

Let's check these assumptions for the lion regression model:

Assume that **error** ? **is** $e_i = y_i - \hat{y}_i$

- normally distributed for each x_i
- has the same variance
- has a mean of 0 at each xi



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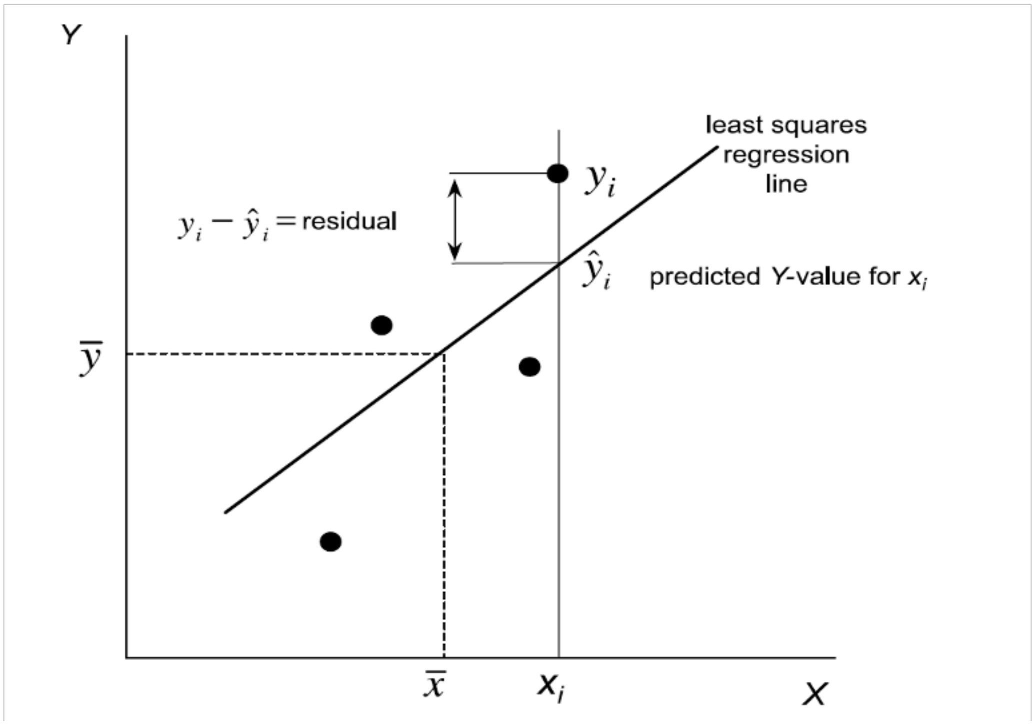
Let's check these assumptions for the lion regression model:

Assume that **error ? is** - estimated as the residuals: $e_i = y_i - \hat{y}_i$

- ordinary lease square estimates a and b or slope and intercept to minimize the sum of the residuals squared or Mean Squared Error (MSE) as

$\sum_{i=1}^{n} = (y_i - \hat{y}_i)^2$



Lecture 9: Linear Regression

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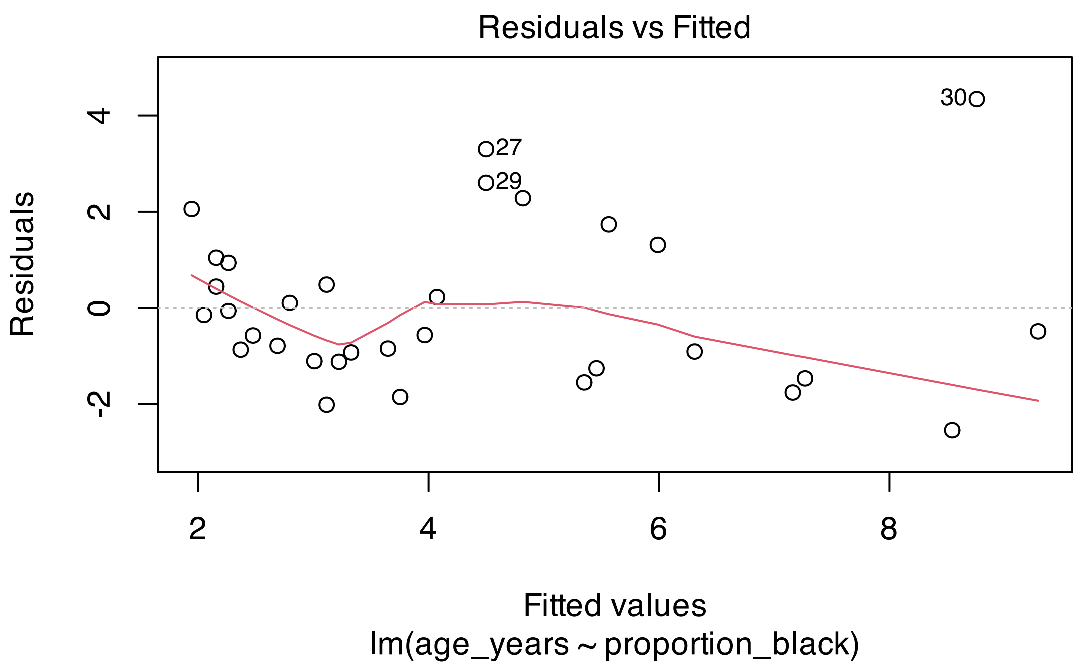

Lecture 9: Linear Regression

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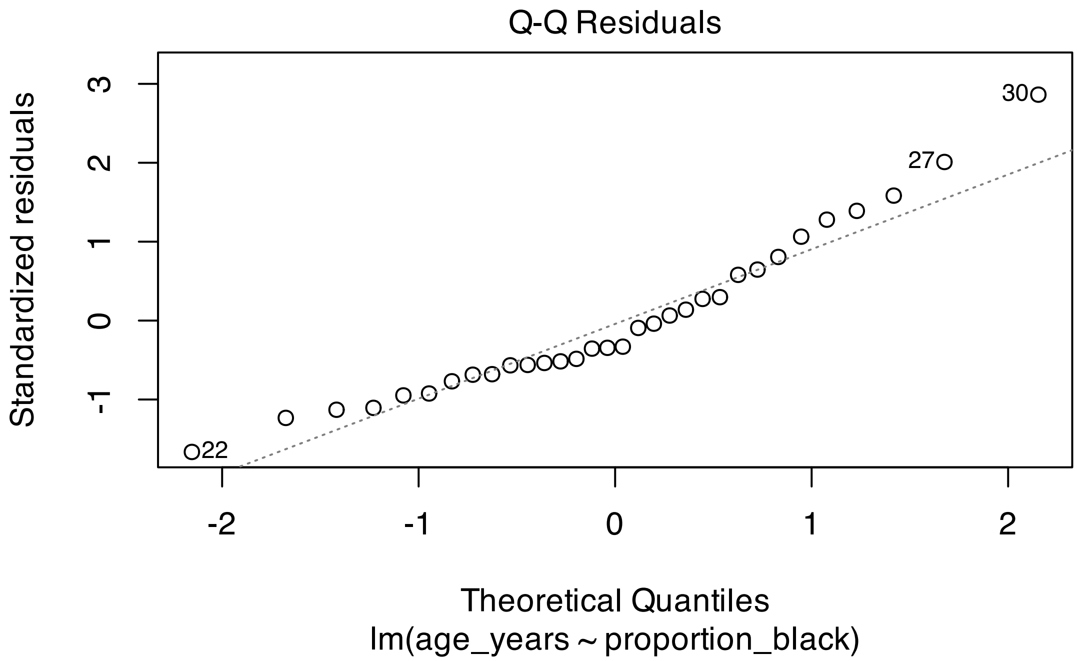

Lecture 9: Linear Regression

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4. **Normality**: Residuals are normally distributed

Let's check these assumptions for the lion regression model:

```
	Shapiro-Wilk normality test

data:  residuals(lion_model)
W = 0.93879, p-value = 0.0692
```

Lecture 9: Linear Regression

Simple Linear Regression Model

linear regression has four key assumptions:

1. **Linearity**: The relationship between X and Y is linear
2. **Independence**: Observations are independent
3. **Homoscedasticity**: Equal variance across all values of X
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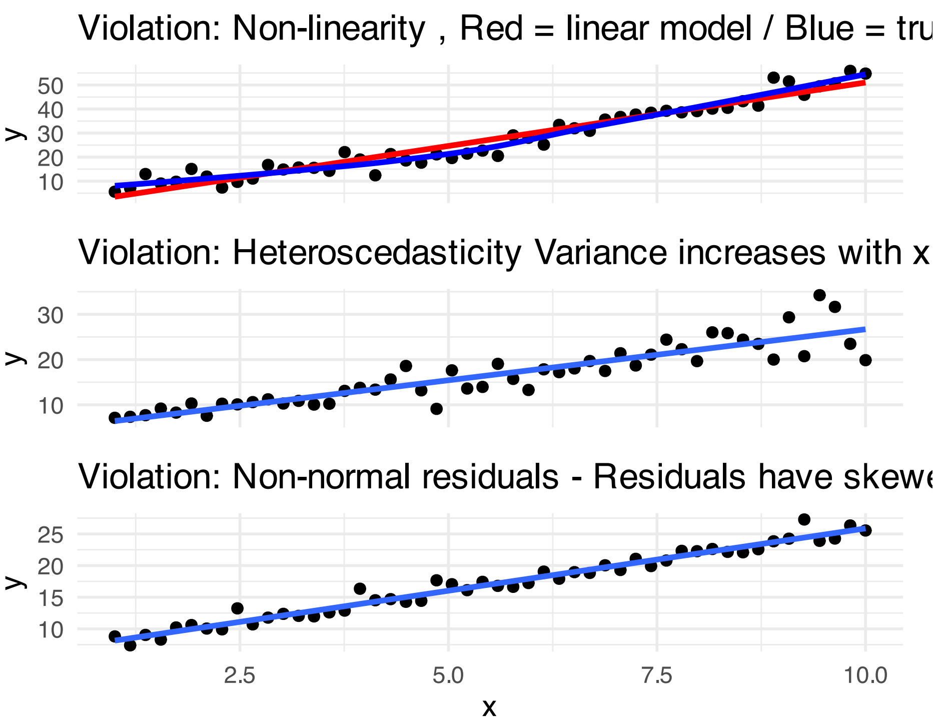
If assumptions are violated: 1. Transform the data (Section 17.6) 2. Use weighted least squares for heteroscedasticity 3. Consider non-linear models (Section 17.8)



Lecture 9: Linear Regression - estimates of error and significance

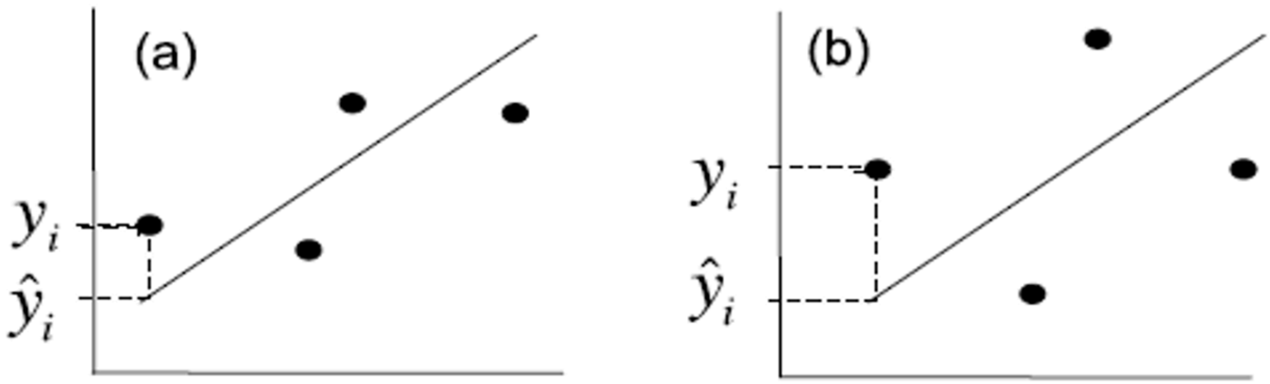
- Estimates of standard error and confidence intervals for slow and intercept to determine confidence bands
- the 95% confidence band will contain the true population line 95/100 under repeated sampling
- this is usually done in R

Parameter	OLS estimate	Standard error
β_1	$b_1 = \frac{\sum_{i=1}^{n}[(x_i-\bar{x})(y_i-\bar{y})]}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$	$s_{b_1} = \sqrt{\frac{MS_{Residual}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$
β_0	$b_0 = \bar{y} - b_1\bar{x}$	$s_{b0} = \sqrt{MS_{Residual}\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]}$
ε_i	$e_i = y_i - \hat{y}_i$	$\sqrt{MS_{Residual}}$ (approx.)

Lecture 9: Linear Regression - estimates of error and significance

In addition to getting estimates of population parameters (β0 , β1), want to test hypotheses about them

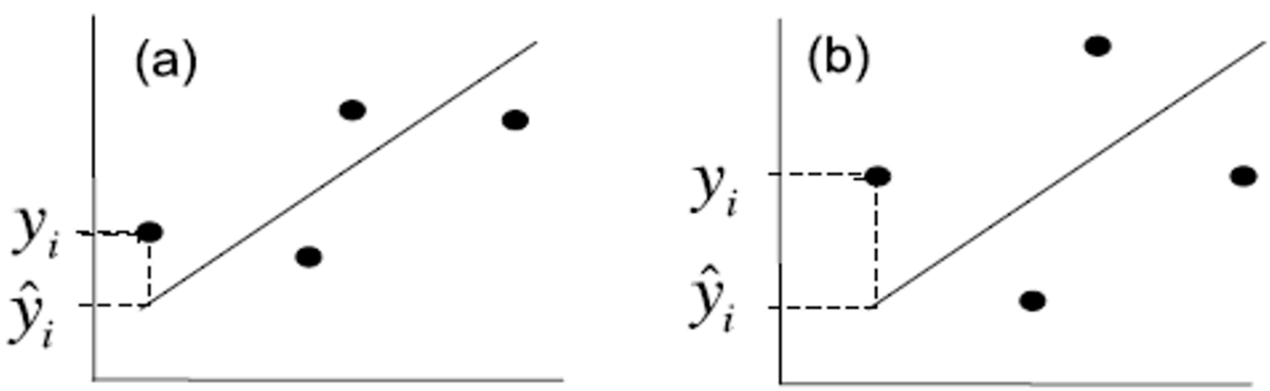
- This is accomplished by analysis of variance
- Partition variance in Y: due to variation in X, due to other things (error)



Lecture 9: Linear Regression - estimates of variance

Total variation in Y is “partitioned” into 3 components:

- $SS_{regression}$: variation explained by regression
  - difference between predicted values ($\hat{y}$i) and mean y ($\bar{y}$)
  - dfs= 1 for simple linear (parameters-1)
- $SS_{residual}$: variation not explained by regression
  - difference between observed (y_i) and predicted ($\hat{y}_i$) values
  - dfs= n-2
- SS_{total}: total variation
  - sum of squared deviations of each observation (y_i) from mean ($\bar{y}$)
  - dfs = n-1



Lecture 9: Linear Regression - estimates of variance

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  - dfs= n-2
- SS_{total}: total variation
  - sum of squared deviations of each observation (y_i) from mean ($\bar{y}$)

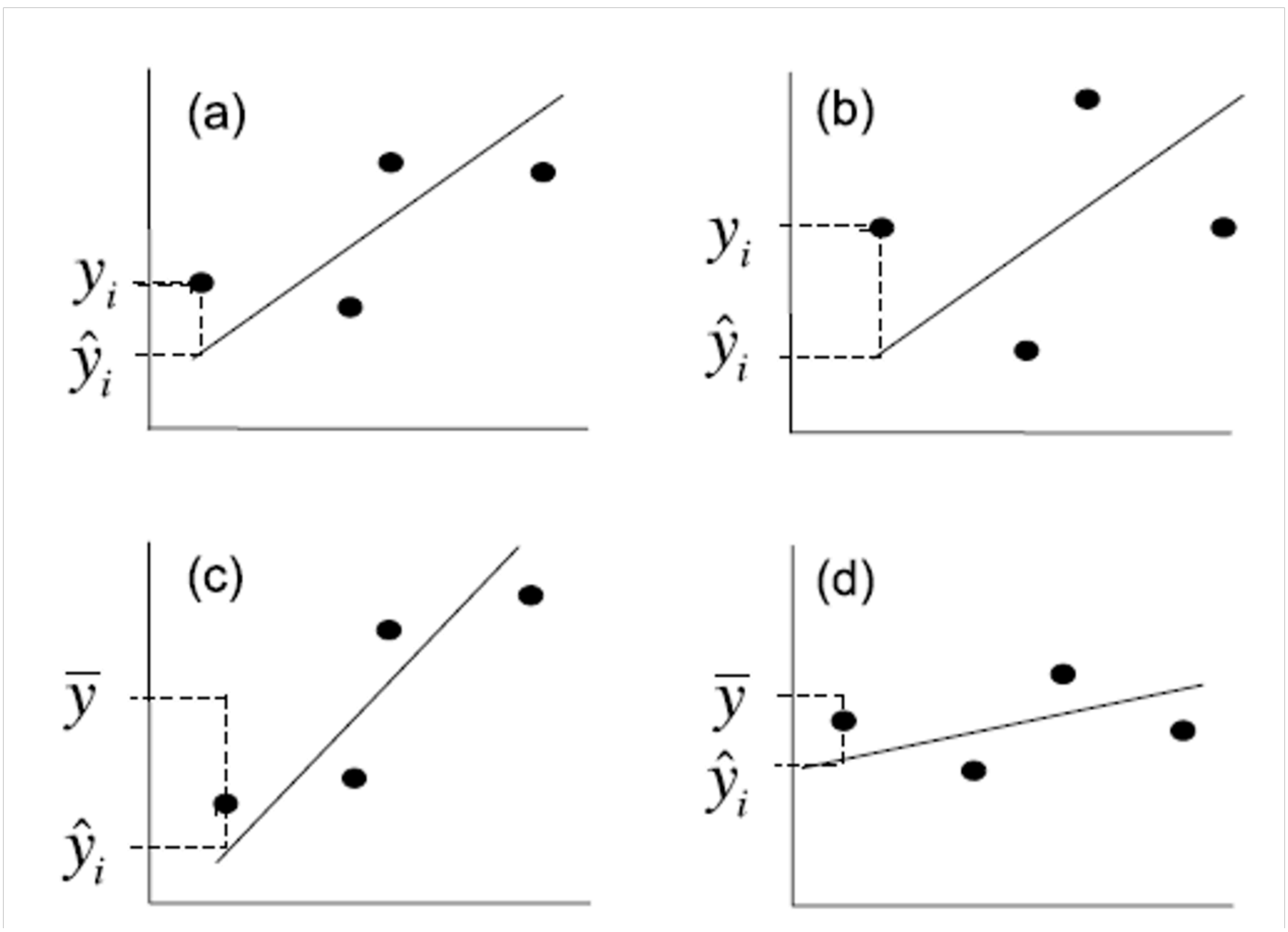
- dfs = n-1

Source of variation	SS	df	MS	Expected mean square
Regression	$\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$	1	$\frac{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{1}$	$\sigma_\varepsilon^2+\beta_1^2\sum_{i=1}^{n}(x_i-\bar{x})^2$
Residual	$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$	$n-2$	$\frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n-2}$	σ_ε^2
Total	$\sum_{i=1}^{n}(y_i-\bar{y})^2$	$n-1$		

Lecture 9: Linear Regression - estimates of variance

Total variation in Y is "partitioned" into 3 components:

- $SS_{regression}$: variation explained by regression
  - GREATER IN C than D
- $SS_{residual}$: variation not explained by regression
  - GREATER IN B THAN A
- SS_{total}: total variation



Lecture 9: Linear Regression - estimates of variance

Sums of Squares and degress of freedome are:

$SS_{regression} + SS_{residual} = SS_{total}$

$df_{regression} + df_{residual} = df_{total}$

- Sums of Squares depends on n
- We need a different estimate of variance

Source of variation	SS	df	MS	Expected mean square
Regression	$\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$	1	$\frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{1}$	$\sigma_\varepsilon^2 + \beta_1{}^2 \sum_{i=1}^{n}(x_i - \bar{x})^2$
Residual	$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$	$n-2$	$\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2}$	σ_ε^2
Total	$\sum_{i=1}^{n}(y_i - \bar{y})^2$	$n-1$		

Lecture 9: Linear Regression - estimates of variance

Sums of Squares converted to Mean Squares

- Sums of Squares divided by degrees of freedom - does not depend on n
- $MS_{residual}$: estimate population variation
- $MS_{regression}$: estimate pop variation and variation due to X-Y relationship
- Mean Squares are not additive

Source of variation	SS	df	MS	Expected mean square
Regression	$\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$	1	$\frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{1}$	$\sigma_\varepsilon^2 + \beta_1{}^2 \sum_{i=1}^{n}(x_i - \bar{x})^2$
Residual	$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$	$n-2$	$\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2}$	σ_ε^2
Total	$\sum_{i=1}^{n}(y_i - \bar{y})^2$	$n-1$		